Elements of Tracking Useful to 3-d Imaging

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Rev. March 27, 2008

1 Track Parameterization

In a recent note [1], we suggested using the parameters \{\phi, \cot \theta, b, z\} to describe detected muon tracks; \phi is the azimuth of the track, \theta is the zenith angle, \(b\) is the impact-parameter, the distance of the point-of-closest-approach of the track to the detector axis, and \(z\) is the height of the track at the point-of-closest-approach. Only two of these parameters (\(\cot \theta, b\)) need to be corrected for the non-zero thickness of the tracking layers.

Measurement errors on track trajectories, which arise from the finite granularity of the tracking system, are estimated in [2]. For the choice of track parameters used here, the corresponding RMS measurement errors are given by:

\[
\begin{align*}
\sigma^2_\phi &= \frac{1}{2} \langle \delta^2 \rangle \\
\sigma^2_{\cot \theta} &= \frac{b^2 \cot^2 \theta \langle \delta^2 \rangle + \langle \zeta^2 \rangle}{2 (R_0^2 - b^2)} \\
\sigma^2_b &= \frac{R_0^2 - b^2}{2} \langle \delta^2 \rangle \\
\sigma^2_z &= \frac{1}{2} \langle \zeta^2 \rangle
\end{align*}
\]

where \(R_0\) is the mean radius of the tracking system and the quantities \(\langle \delta^2 \rangle, \langle \zeta^2 \rangle\) are the variances in the azimuth and longitudinal position, respectively, of the entrance and exit locations of measured tracks. In the approximation that the two stereo wrap-angles are exactly \(\pi\) and the stereo angle of each is exactly 30 degrees, the hit-location variances are given by:

\[
\langle \delta^2 \rangle = \frac{w^2}{30 R_0^2}
\]
\[ \langle \zeta^2 \rangle = \frac{w^2}{6} \]  

where \( w \) is the width of the scintillator strips. For the present detector, \( w = 30 \text{ mm} \) and \( R_0 = 800 \text{ mm} \), in which case the RMS measurement errors for the specified set of track parameters are:

\[
\sigma_\phi \approx \frac{1}{206} \\
\sigma_{\cot \theta} \approx \frac{1}{92} \left( 1 + \frac{b^2 \cot^2 \theta}{5 R_0^2} \right)^{1/2} \\
\sigma_b \approx 3.9 \text{ mm} \left( 1 - \frac{b^2}{R_0^2} \right)^{1/2} \\
\sigma_z \approx 8.7 \text{ mm}
\]

These resolutions suggest we use 1300 bins in \( \phi \) to cover the range \( 0 - 2 \pi \) with 1-\( \sigma \) bins and 230 bins in \( \cot \theta \) to cover the range \( 0 - 2.5 \). The resolutions on track location at the detector are much less than the size of objects of interest in muon tomography; the physical scale of objects of interest should be considered when choosing bin sizes for histogramming these track parameters. A resolution of 0.5 cm for impact parameter and 1 cm for the \( z \)-location represent reasonable choices.

## 2 Intersection of a Track with a Plane

As described in reference [2], the trajectory of a given track is conveniently described by the two vectors \( \hat{u} \), a unit-vector pointing opposite to the direction of the muon track, and \( \mathbf{X}_t \), the location of the point-of-closest-approach to the detector axis. These vector quantities are related to the four basic track parameters by:

\[
\hat{u} = \frac{\hat{\rho}(\phi) + \cot \theta \hat{z}}{(1 + \cot^2 \theta)^{1/2}} \\
\mathbf{X}_t = -b \hat{\phi}(\phi) + z \hat{z}
\]

where the (cylindrical-basis) unit-vectors \( \hat{\rho}, \hat{\phi} \) are defined by:
\[ \hat{\rho}(\phi) = \cos \phi \hat{x} + \sin \phi \hat{y} \quad (13) \]
\[ \hat{\phi}(\phi) = -\sin \phi \hat{x} + \cos \phi \hat{y} \quad (14) \]

Planes can be described by a unit-vector \( \hat{n}_p \) normal to the plane and by the location \( X_p \) of the origin of coordinates in the plane relative to the detector center. With these definitions, the distance \( d \) from the origin of coordinates in the plane to the intersection point is given by:
\[
d = \hat{n}_p \times [(X_t - X_p) \times \hat{u}] / \hat{n}_p \cdot \hat{u} \quad (15)
\]

### 2.1 Vertical Planes

Many useful cases are vertical planes which can be described by \( X_p = R \hat{\rho}(\Phi) + H \hat{z} \), \( \hat{n}_p = -\hat{\rho}(\Phi + \alpha) \). Local horizontal and vertical directions in a vertical plane can be defined by \( \hat{h} = -\hat{\phi}(\Phi + \alpha) \), \( \hat{v} = \hat{z} \). \( \alpha \) describes the orientation of the plane relative to the local radial direction. The local coordinates of the track intersection are:
\[
d_h = \hat{h} \cdot d = \frac{b + R \sin(\Phi - \phi)}{\cos(\Phi + \alpha - \phi)} \quad (16)
\]
\[
d_v = \hat{v} \cdot d = z - H + \cot \theta \left[ \frac{R \cos \alpha + b \sin(\Phi + \alpha - \phi)}{\cos(\Phi + \alpha - \phi)} \right] \quad (17)
\]

### 2.2 Horizontal Planes

The intersection of a track with a horizontal plane located a height \( H \) above the center of the detector can be described in the \( x - y \) coordinate system of the detector:
\[
x = \left( \frac{H - z}{\cot \theta} \right) \cos \phi + b \sin \phi \quad (18)
\]
\[
y = \left( \frac{H - z}{\cot \theta} \right) \sin \phi - b \cos \phi \quad (19)
\]

### 2.3 Planes Locally Tangent to a Sphere

Planes locally tangent to a sphere centered on the detector origin are described by \( X_p = R[\cos \lambda \hat{\rho}(\Phi) + \sin \lambda \hat{z}] \), \( \hat{n}_p = -[\cos \lambda \hat{\rho}(\Phi) + \sin \lambda \hat{z}] \), where
$R$ is the radius of the sphere, $\Phi$ is the azimuth of the origin and tangent-point of the plane, and $\lambda$ is the “latitude” of the origin/tangent-point. The local horizontal and vertical directions in the tangent plane are $\hat{h} = -\hat{\phi}(\Phi)$, $\hat{v} = \cos \lambda \hat{z} - \sin \lambda \hat{\rho}(\Phi)$. The local coordinates of the track intersection are:

\[
\begin{align*}
d_h &= \frac{R \sin(\Phi - \phi) + b [\cos \lambda + \sin \lambda \cot \theta \cos(\Phi - \phi)] - z \sin \lambda \sin(\Phi - \phi)}{\sin \lambda \cot \theta + \cos \lambda \cos(\Phi - \phi)} \\
d_v &= \frac{R [\cos \lambda \cot \theta - \sin \lambda \cos(\Phi - \phi)] + b \cot \theta \sin(\Phi - \phi) + z \cos(\Phi - \phi)}{\sin \lambda \cot \theta + \cos \lambda \cos(\Phi - \phi)}
\end{align*}
\]

### 3 Correlations Among Track Parameters

Detected tracks that pass through a common volume-element—voxel—have track parameters that are correlated by the geometry of the problem. These correlations are exploited in image reconstruction algorithms (such as Radon transforms) to determine locations of voxels of higher or lower density. Such correlations are apparent in Fig. 1 where different tracks (labeled by $i$) have intersections with the detector ($b_i$ and $z_i$) that are correlated with track slopes in the horizontal and vertical directions ($\sin(\Phi - \phi_i)$ and $\cot \theta_i$); unfolding these correlations can determine the position of the voxel ($R$, $\Phi$, $H$).

Equations 16, 17 demonstrate the correlations between track parameters. Taking the voxel of interest to have coordinates $\{R, \Phi, H\}$, which corresponds to setting $d_h = 0$, $d_v = 0$ in expressions above, gives the following relations among parameters:

\[
\begin{align*}
b_i &= R \sin(\phi_i - \Phi) \\
z_i &= H - R \cot \theta_i \cos(\phi_i - \Phi)
\end{align*}
\]

Relations 20 and 21 suggest a simple algorithm for finding voxels of interest in the data:

1. From a histogram of $(\phi, \cot \theta)$ or other information, determine azimuthal locations of interest, $\Phi$.
2. Histogram $(b, \sin(\phi - \Phi))$ for $\phi$ in the vicinity of $\Phi$; a correlation gives information on $R$.
3. Histogram $(z, \cot \theta \cdot \cos(\phi - \Phi))$ for $\phi$ in the vicinity of $\Phi$; the slope of a correlation gives $R$, the intercept gives $H$. 


Figure 1: Illustration of correlations between track parameters for a track, labeled by $i$, which passes through a particular voxel—indicated by the black cube—located a distance $R$ from the detector axis at height $H$ above the detector mid-plane and at the azimuth $\Phi$. a) Horizontal plane, described by Eqn. 20; b) Vertical plane, described by Eqn. 21.
4 Conclusions

Bin sizes for histograms of track parameters used to form images should generally be chosen comparable to expected resolutions for determining the parameters. Thus, the “standard” \((\phi, \cot \theta)\) histogram should have 1300 bins in \(\phi\) over the range \(0 - 2\pi\) and 230 bins in \(\cot \theta\) over the range \(0 - 2.5\).

It is not practical to maintain 4-dimensional histograms for all track parameters over their complete ranges with bin sizes comparable to the respective resolutions. Rather, files of the individual track parameters should be produced and archived. The relevant track parameters can be histogrammed when specific 3-dimensional regions of interest are determined, usually by specifying a particular sub-range for \(\phi\). In such histograms, the bin size can be chosen to be comparable to the resolution of the parameter of choice. Thus, for example, when \(b\) is being histogrammed, we should use 256 bins over the range \(-64\text{ cm} - +64\text{ cm}\). When \(z\) is being histogrammed, we should use 420 bins over the range \(-210\text{ cm} - +210\text{ cm}\). In the histograms for determining voxel coordinates over selected regions of azimuth, the bin sizes for \(\phi\) (or \(\sin(\phi - \Phi)\)) and \(\cot \theta\) should remain about the same as the values used in the standard \((\phi, \cot \theta)\) histogram: \(\Delta \phi \approx 0.005\), \(\Delta \cot \theta \approx 0.010\).

References
